S. Keel II

L lattice $= \mathbb{Z}^N$, $T_L = L \otimes \mathbb{C}^*$ $\qquad$ L = cocharacters of T_L
$\qquad$ L^* = characters of T_L

$G \in L^* \rightsquigarrow z^G : T_L \to \mathbb{C}^*$ function = monomial in coordinates.

Basic mutation:

Take $v \in L$, $F \in L^*$ st. $F(v) = 0$.

$\to m = m_{(v,F)} : T_L \dashrightarrow T_L$ birational, $m^*(z^G) = z^G(1+z^F)^{G(v)}$ $\forall G$.

$\hookrightarrow$ induces iso $m^* : Q \to Q$ field of rat'l functions.

satisfies $m^*(\omega) = \omega$

Induces $m^t_{(v,F)} : \underset{L}{\overset{\shortparallel}{T_L^{trop}}} \ni v \dashrightarrow \underset{L}{\overset{\shortparallel}{T_L^{trop}}}$ $\qquad$ $v \mapsto v \circ m^*$, ie.

Think of T_L^{trop} = valuations $v : Q(T_L) \setminus \{0\} \xrightarrow{v} \mathbb{Z}$

$$\begin{tikzcd} Q(T_L)\setminus\{0\} \arrow[r, "v"] & \mathbb{Z} \\ Q(T_L)\setminus\{0\} \arrow[u, "m^*", "\simeq"'] \arrow[ur, "v \circ m^*"'] & \end{tikzcd}$$

Extends to $m^t : L_{\mathbb{R}} \to L_{\mathbb{R}}$ PL transformation

compute $\Rightarrow$ find that $m^t : x \mapsto x + \underbrace{\max(F(x), 0)}_{=: F(x)_+} v$

ie. m^t:

$F<0$	$F>0$
Id	$x \mapsto x + F(x)v$

($F=0$ dividing line; $\mathbb{R}^n$)

Very simple CY: $N = \mathbb{Z}^N$ $\rightsquigarrow$ take copies of torus T_N, $\quad T_{N,s}$ $s \in I$ $\uparrow$ set of seeds for mutation.

$\Rightarrow \mathcal{A} = \bigcup_{\substack{s \in I \\ \text{seeds}}} T_{N,s}$ glued by

$m_{s,s'} : T_{N,s} \dashrightarrow T_{N,s'}$ of the sort above.

$\mathcal{A}$ = CY with maximal ∂.

Each seed gives a tropicalization $\mathcal{A}^t(\mathbb{R}) = T^{trop}_{N,s}(\mathbb{R}) = N_{\mathbb{R}}$.
w/ PL structure

These come in dual pairs: if $M = N^*$, can do simultaneously $m_{(v,F)} : T_{N,s} \dashrightarrow T_{N,s'}$ and $m_{(F,v)} : T_{M,s} \dashrightarrow T_{M,s'}$

$\rightsquigarrow$ Fock–Goncharov dual $\mathcal{X} = \bigcup_{s \in I} T_{M,s}$ vs $\mathcal{A} = \bigcup_{s \in I} T_{N,s}$

Naive hope: $\mathcal{X}$ and $\mathcal{A}$ are mirror dual.

Eg. hope: $\mathcal{A}^{trop}(\mathbb{Z}) \longrightarrow$ basis of $\Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$

$\mathcal{X}^{trop}(\mathbb{Z}) \to$ basis of $\Gamma(\mathcal{A}, \mathcal{O}_{\mathcal{A}})$

Thm: "this works"

$\star$ U CY with max. ∂ (eg. $\mathcal{A}$ or $\mathcal{X}$ above), $B = U^{trop}(\mathbb{R})$ $(= PL\ \mathbb{R}^N)$

Discussed last time: algebra structure on $V = \bigoplus_{q \in B(\mathbb{Z})} k \cdot \theta_q$

Describe conjecturally the mult. rule directly

$$\theta_p \cdot \theta_q = \sum_{r \in B(\mathbb{Z})} \alpha(p, q, r)\, \theta_r$$

Note: straight lines in $B(\mathbb{R})$ don't make sense (PL only).

Main conj: $\exists$ canonical piecewise straight paths = "broken paths" (or "trop. disks")

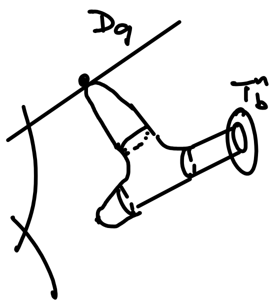
can be drawn for each seed

(eg. in $N_{s,\mathbb{R}} = B(\mathbb{R}) = \mathbb{R}^N$ $\forall s$ seed in $\mathcal{A}$-case)

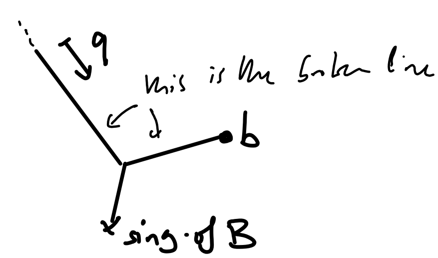
ie. notion of broken straight line is invt under PL mutations.

Morally: tropicalize discs in base B of SYZ fibration:

$U \subset Y$



$\rightsquigarrow$ $\begin{array}{c} U \\ \pi \downarrow \\ B \ni b \end{array}$ fiber $T^N = (S^1)^N$

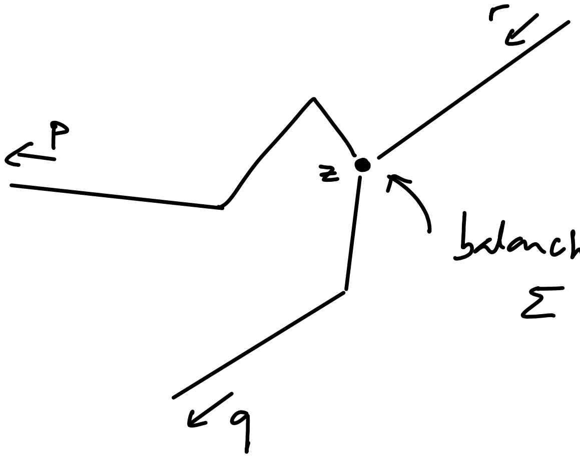


we only draw the path from ∞ to b $\rightsquigarrow$ "broken line" γ

Say γ goes from q to b.

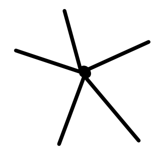
(all slopes are rational)

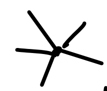
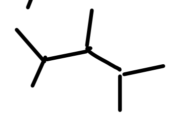
$\theta_p \theta_q = \sum \alpha(p,q,r)\, \theta_r$: $\alpha(p,q,r)$ counts pictures =



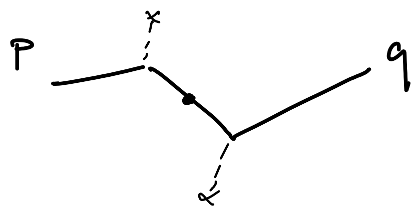
$(p, q, r \in T_z B)$

balancing condition
$\sum$ tangent vectors $= 0$

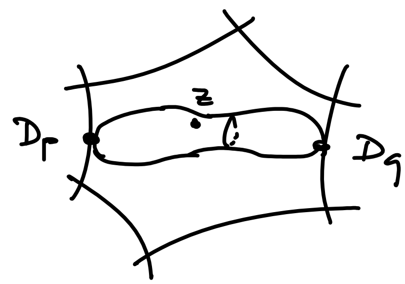
<u>NB</u>: higher product = similarly w/ more legs out of vertex 
(still $\sum$ tangent vectors $= 0$)

[cross-ratio fixing cond. in log GW interpretation explains why 
and not ]

<u>Eg.</u> for $r = 0$, coeff of θ_0 in $\theta_p \theta_q$ = count



$\Longleftrightarrow$

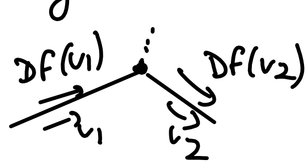


• Since want a mult. rule on $V = \bigoplus_q \theta_q k$, need <u>finiteness</u> property?
to define CY Spec V

Notion of <u>min-convexity</u> along broken lines:

<u>Def</u>: f : $B \to \mathbb{R}$ piecewise linear is "min-convex" if, along any broken line, f is "min-convex" ie. its derivative $\Downarrow$

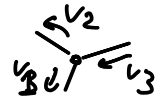
Eg. a min of linear functions is min-convex on usual $\mathbb{R}^n$ w/ straight lines

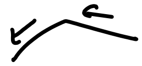
Here need 

$Df(v_1) \geq Df(v_2)$

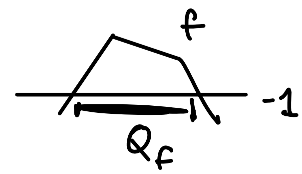
["min-convex" is usually called "concave"...]

Suppose f is min-convex & have picture so ϑ_r appears in $\vartheta_p \cdot \vartheta_q$
$\Rightarrow$ can check $f(r) \geq f(p) + f(q)$

(since at  $Df(v_3) = Df(v_1) + Df(v_2)$

and at breaks  $Df \Downarrow$

So, now, let $Q_f = \{ q : f(q) \geq -1 \}$ "convex polygon" in B



If Q_f is <u>bounded</u> then set of asymptotic directions r in which $f(r) \geq f(p) + f(q)$ is a <u>finite</u> subset of $B_{\mathbb{Z}}$ for given p, q.

... multiplication rule is then polynomial!

<u>Conj</u> (<u>Thm</u> in dim. 2 [T. Mandel])

If $\varphi : U \to \mathbb{A}^1$ is a regular function, then $\varphi^{trop} : B \to \mathbb{R}$ is min-convex, where $\varphi^{trop}(q) = val_{D_q}(f)$.

$\forall$ min-cvx fn, get a filtration on V ... let $\tilde{V} \subset V[T]$ subspace gen. by $\vartheta_q T^s$

$\rightsquigarrow$ $\mathrm{Spec}(V) \subset \mathrm{Proj}(\tilde{V})$ if Q_f is bounded
$\quad$ "
Mirror to U

<u>Conj</u> This is a minimal model for $U^\vee$ and all minimal models arise in this manner.

<u>Thm:</u> In cluster case, if $\exists f$ with Q_f bounded then $\mathrm{Spec}(V)$ is affine CY w/ maximal ∂, and $\mathrm{Proj}(V)$ is a minimal model for it.

Each choice of seed s gives $Q_f \subset B(\mathbb{R}) = N_{s,\mathbb{R}} = \mathbb{R}^N$

(a different polytope! they change by PL transformations) upon mutation
for each s

$\rightarrow$ get a <u>canonical degeneration</u> of $\mathrm{Proj}(\tilde{V})$ to $\mathrm{ToricVar}(Q_{f,s})$ assoc. to this choice of seed s

$\mathcal{A} \longleftrightarrow$ basis of functions on $\mathcal{X}^{trop}(\mathbb{R})$

take φ on $\mathcal{A}$, $\varphi^{trop}: \mathcal{A}^{trop} \to \mathbb{R}$ min-convex

$$Q_\varphi = \{\varphi^{trop} \geq -1\}$$

$$\rightsquigarrow \operatorname{Proj}(\widetilde{V}_\varphi) \underset{\text{minimal model}}{\supset} \operatorname{Spec}(V) = \mathcal{X}.$$

<u>Conj</u>: θ-functions appearing on $\varphi = c_1\theta_{x_1} + \dots + c_s\theta_{x_s}$ give a subset of $\mathcal{X}^{trop}(\mathbb{Z})$